The Numbers of the Icosahedron — An Essay on φ, sin 54°, and sin 72°

1. Introduction

The regular icosahedron is a Platonic solid made of 20 equilateral triangular faces. Once a single edge length ℓ is fixed, every other measurement of the shape is determined — and in computing those measurements, numbers like 0.618, 0.809, 0.951, and 1.618 keep reappearing. This essay derives, from scratch, using nothing beyond high-school mathematics (the angle sum of a triangle, similarity, the quadratic formula, and trigonometric ratios), what these numbers are and why they take exactly these values.

2. The basic numbers: 20, 30, 12

- Faces: **20** equilateral triangles
- Edges: **30**
- Vertices: **12**

These three numbers satisfy Euler's polyhedron formula:

$$V - E + F = 12 - 30 + 20 = 2$$

One more combinatorial fact worth holding onto: **exactly 5 triangles meet at every vertex.** This "5" is the root of every number that follows.

3. The golden ratio φ: an algebraic definition

The golden ratio is defined as the positive root of $x^2 = x + 1$:

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

Nothing in this definition mentions an angle or a polyhedron — it is a purely algebraic number. Yet it turns up everywhere once you start computing the icosahedron's dimensions. Tracing why requires starting from one particular triangle.

4. The golden triangle: 36°-72°-72°

4.1 Why the base angles are 72°

In an isosceles triangle, the only choice we make is the apex angle: **36°**. The base angles follow automatically from the angle sum of a triangle, 180°.

$$36^\circ + \text{base angle} \times 2 = 180^\circ \quad\Longrightarrow\quad \text{base angle} = \frac{144^\circ}{2} = 72^\circ$$

The apex is set to 36° because that angle comes from the regular pentagon. A regular pentagon's central angle is 360° ÷ 5 = 72°, and the sharp point of a pentagram (five-pointed star) is exactly half that, 36°. If you take one side of a regular pentagon as the base and its two diagonals as the equal sides, this 36°-72°-72° triangle appears directly.

4.2 Defining the side lengths and setting up the equation

Let the two equal sides have length 1, and let the base be b. The figure below shows this setup and the next step.

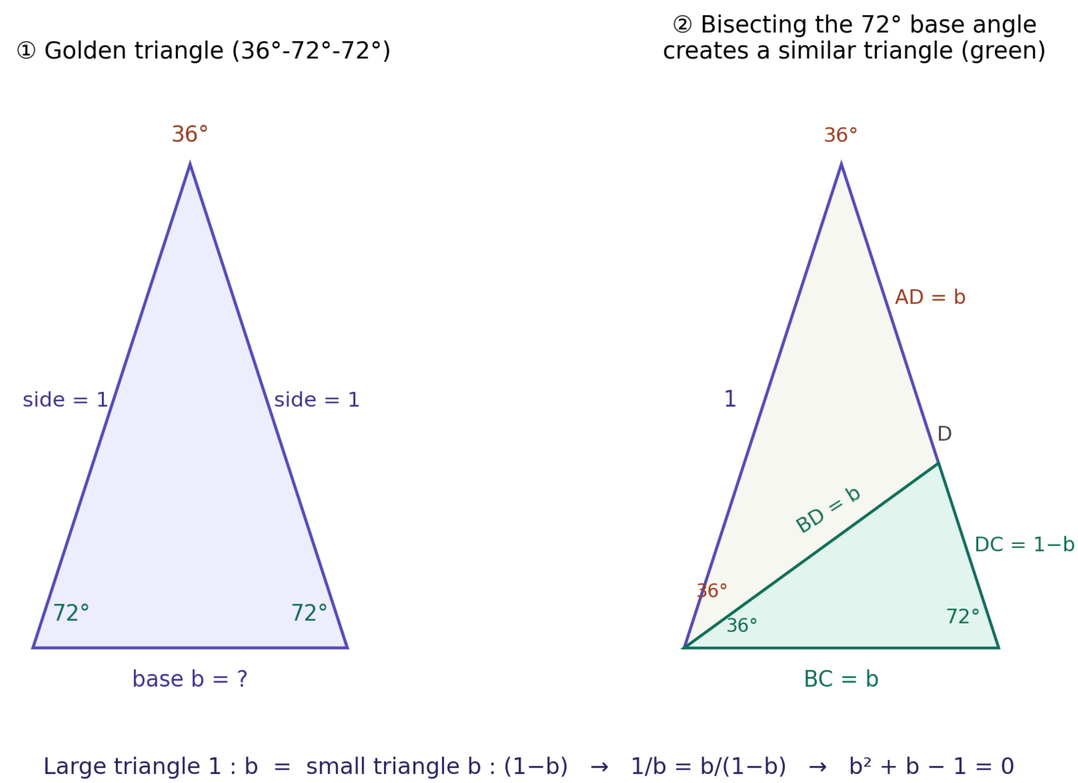

Figure 1: Golden triangle and bisection of the base angle

The left panel (①) is the starting point: equal sides AB = AC = 1, base BC = b (unknown). In the right panel (②), the base angle at B (72°) is bisected. Let D be the point where the bisector meets side AC. The side lengths are then fixed as follows:

- **Triangle BDC**: angle DBC = 36°, angle C = 72°, so angle BDC = 72°. This is a 36°-72°-72° triangle — **similar to the original**. Since it is isosceles, the sides opposite equal angles (72°) are equal: **BD = BC = b**
- **Triangle ABD**: angle A = 36°, angle ABD = 36°, so it too is isosceles. Hence **AD = BD = b**
- Therefore **DC = AC − AD = 1 − b**

Now read the ratio (equal side) : (base) from the two similar triangles. The large triangle gives 1 : b; the small triangle (BDC) gives b : (1−b). Similarity means the two ratios are equal:

$$\frac{1}{b} = \frac{b}{1-b}$$

Cross-multiplying (product of the extremes equals product of the means):

$$1 - b = b^2 \implies b^2 + b - 1 = 0$$

4.3 Solving with the quadratic formula

Comparing with the general form $ax^2 + bx + c = 0$ gives a = 1, b (coefficient) = 1, c = −1. Substituting into the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

Since $\sqrt{5} \approx 2.236$, the two roots are 0.618 and −1.618. Since a length must be positive:

$$b = \frac{\sqrt{5} - 1}{2} \approx 0.618$$

Check: $b^2 + b - 1 = 0.382 + 0.618 - 1 = 0$ ✓

4.4 Identifying this value — the reciprocal of the golden ratio

$$\varphi \times \frac{\sqrt{5} - 1}{2} = \frac{(1 + \sqrt{5})(\sqrt{5} - 1)}{4} = \frac{5 - 1}{4} = 1 \implies b = \frac{1}{\varphi}$$

The discarded negative root has magnitude exactly φ. A single bisection of the base angle was enough to produce the golden ratio — which is exactly why this triangle is called the golden triangle.

5. cos 72°

Drop a perpendicular from the apex of the golden triangle to the base. This splits the base into two segments of length b/2 and creates a right triangle containing the 72° base angle. By the definition of cosine (adjacent/hypotenuse):

$$\cos 72° = \frac{b/2}{1} = \frac{1}{2} \cdot \frac{\sqrt{5} - 1}{2} = \frac{\sqrt{5} - 1}{4} = \frac{1}{2\varphi} \approx 0.309$$

6. sin 72°

Substituting into $\sin^2 + \cos^2 = 1$:

$$\sin^2 72° = 1 - \left(\frac{\sqrt{5} - 1}{4}\right)^2 = 1 - \frac{6 - 2\sqrt{5}}{16} = \frac{10 + 2\sqrt{5}}{16}$$

$$\sin 72° = \frac{\sqrt{10 + 2\sqrt{5}}}{4} \approx 0.951$$

7. sin 54° and the golden gnomon

For an isosceles triangle with equal sides ℓ and apex angle θ, the base is $b = 2\ell\sin(\theta/2)$. Now look for the triangle whose base-to-side ratio is the golden ratio itself ($b = \phi\ell$):

$$\sin\frac{\theta}{2} = \frac{\varphi}{2} \approx 0.809 \implies \frac{\theta}{2} = 54°, \quad \theta = 108°$$

A wide isosceles triangle with apex angle 108° — this is the **golden gnomon**. Cutting a regular pentagon along its diagonals produces only two kinds of triangle: the sharp golden triangle and this wide golden gnomon. From here comes a key identity:

$$\frac{\varphi}{2} = \sin 54° = \cos 36° \approx 0.809 \qquad \Longleftrightarrow \qquad \varphi = 2\sin 54°$$

8. The icosahedron's radii

For an icosahedron with edge length ℓ, the characteristic lengths are:

- **Circumradius** (center → vertex): $(\ell/4)\sqrt{10+2\sqrt{5}}$ = **$\ell\sin 72° \approx 0.951\ell$**
- **Midradius** (center → edge midpoint): $(\ell/4)(1+\sqrt{5})$ = **$(\phi/2)\ell = \ell\sin 54° \approx 0.809\ell$**
- Inradius (center → face center): $\approx 0.756\ell$

0.951 and 0.809 are not a contradiction — they are two different answers to two different questions (distance to a vertex versus distance to an edge midpoint). In any calculation, it's essential to first pin down which of these R refers to.

9. The root cause: 72° = 360°/5

Angle	Geometric meaning	Value
72°	Central angle of a regular pentagon (360°/5)	sin 72° ≈ 0.951
36°	Half of 72°, the pentagram's point angle	cos 36° = $\phi/2$ ≈ 0.809
54°	90° − 36°	sin 54° = $\phi/2$ ≈ 0.809
108°	Interior angle of a regular pentagon (180° − 72°)	apex angle of the golden gnomon

Five triangles meet at every vertex of the icosahedron, and the five neighboring vertices of any one vertex form a regular pentagon. This five-fold rotational symmetry is why pentagon angles seep into every dimension of the icosahedron.

5 triangles per vertex → 5-fold symmetry → regular pentagon (72°) → golden triangle & golden gnomon → quadratic formula → ϕ

10. A coordinate check: (0, ±1, ±ϕ)

The icosahedron's 12 vertices are given by the following coordinates and their cyclic permutations:

$$(0,\, \pm 1,\, \pm\varphi), \quad (\pm 1,\, \pm\varphi,\, 0), \quad (\pm\varphi,\, 0,\, \pm 1)$$

These are the vertices of three mutually perpendicular golden rectangles (side ratio 1 : ϕ). In this coordinate system:

- Edge length: the distance between (0, 1, ϕ) and (0, −1, ϕ) is 2
- Circumradius: from the origin to (0, 1, ϕ) is $\sqrt{(1+\phi^2)} \approx 1.902$
- Ratio: 1.902 / 2 ≈ **0.951 = sin 72°** ✓

11. Conclusion

Combinatorially, the icosahedron is 20-30-12 (faces-edges-vertices), and metrically it carries two characteristic lengths in terms of edge ℓ: 0.951ℓ (circumradius, sin 72°) and 0.809ℓ (midradius, sin 54° = ϕ/2). Both values fall out of a single calculation: bisect the base angle of the 36°-72°-72° golden triangle, use similarity to set up $b^2 + b - 1 = 0$, and solve with the quadratic formula to get $b = (\sqrt{5}-1)/2 = 1/\phi$. From there, cos 72° follows, sin 72° follows from the Pythagorean identity, and those values become the polyhedron's radii. The tools used throughout — the angle sum of a triangle, similarity, the quadratic formula, and the definitions of the trigonometric ratios — are all high-school mathematics, and the whole chain starts from a single line: the quadratic formula.